Partial Schur multiplier

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Abstract

The partial Schur multiplier $pM(G)$ of a group G is a generalization of the classical Schur multiplier $M(G)$. While its classical version is a group, $pM(G)$ is a semilattice of abelian groups $pM_D(G)$ (called components), indexed by certain subsets $D \subseteq G \times G$. Each component $pM_D(G)$ consists of partially defined functions $\sigma\colon G \times G \to K$ having D as its domain. Such functions are called partial factor sets of G and are associated to the partial projective representations of G. This work discusses a specific component, $pM_{G\times G}(G)$, which is particularly interesting due to the fact that there are epimorphisms from it to any other component, assuming that K is an algebraically closed field. We characterize this component for some families of groups, including the dihedral and dicyclic groups. This is a joint work with H. Pinedo.

1 Preliminaries

In [1] it was obtained the following decomposition for the partial Schur multiplier:

Theorem 1. *The semigroups $pm(G)$ and $pM(G)$ are semilattices of abelian groups*

$$pm(G) = \bigcup_{D\in C(G)} pm_D(G) \quad \text{and} \quad pM(G) = \bigcup_{D\in C(G)} pM_D(G),$$

where $C(G)$ is a semilattice subsets of $G \times G$, and the factor sets in $pm_D(G)$ have domain D.

The semilattice $(C(G), \cap)$ is formed by the subsets of $G \times G$ which are invariant under the action of a specific semigroup $\mathcal{T}$ (it is the same for every group G), which is generated by symbols u, v and t with relations

$$u^2 = v^2 = (uv)^3 = 1,\ t^2 = t,\ ut = t,\ tuvt = tvuv,\ tvt = 0.$$

This semigroup $\mathcal{T}$ contains a copy $\mathcal{S} = \langle u, v \mid u^2 = v^2 = (uv)^3 = 1\rangle$ of the symmetric group S_3, and acts on $G \times G$ by $t(x, y) = (x, 1),\ u(x, y) = (xy, y^{-1})$ and $v(x, y) = (y^{-1}, x^{-1})$.

The orbits of the action of S_3 may have 1, 2, 3, or 6 elements. An S_3 orbit with 2 or 6 elements will be called effective.

If K is an algebraically closed field, it is known from [3] that each element of $pM(G)$ has some representative which is determined by its values on these effective orbits. More specifically, each component $pm_D(G)$ of $pm(G)$ has a subgroup $pm'_D(G)$, formed by the maps $\sigma\colon G \times G \to K^*$ satisfying

$$\sigma(a, b)\sigma(b^{-1}, a^{-1}) = 1_K,$$
$$\sigma(a, b) = \sigma(b^{-1}a^{-1}, a) = \sigma(b, b^{-1}a^{-1}),$$
$$\sigma(a, 1) = 1_K,$$

and a factor set $\sigma \in pm'_D(G)$ is completely determined by its values in the representatives of the effective S_3-orbits of D.

Lemma 1. *If $\sigma \in pm'_{C_m\times C_m}(C_m)$ and $m \geq 3$ then σ is uniquely determined by its values in the elements from*

$$S_{C_m} = \{(a^i, a^j) \mid 1 \leq i \leq \lfloor(m-1)/3\rfloor \text{ and } i \leq j \leq m-2i-1\} \cup \{(a^i, a^j) \mid i = j = m/3 \in \mathbb{Z}\}.$$

Moreover, these values can be chosen arbitrarily in K^*.

Lemma 2. *Let $G = C_m \times C_2$ or $G = D_{2m}$, and $m \geq 3$. Then, with the notation from Lemma 1, any $\sigma \in pm'_{G\times G}(G)$ is completely determined by its values on the elements from*

$$S_G = S_{C_m} \cup \{(a^k, a^l b) \mid 1 \leq k \leq \lfloor(m-1)/2\rfloor \text{ and } 0 \leq l \leq m-1\} \cup \{(a^k, a^l b) \mid \text{if} \quad k = m/2 \in \mathbb{Z} \text{ and } 0 \leq l \leq (m/2)-1\}.$$

Moreover, these values can be chosen arbitrarily in K^*.

2 Dihedral groups

On [6] it was proved that the total component $pM_{S_3\times S_3}(S_3)$ is isomorphic to $(K^*)^3$. We obtained the following generalization for dihedral groups $D_{2m} = \langle a, b \mid a^m = b^2 = (ab)^2 = 1\rangle, m \in \mathbb{N}$:

Theorem 2. *$pM_{D_{2m}\times D_{2m}}(D_{2m}) \simeq (K^*)^{d_m - \lfloor\frac{m-1}{2}\rfloor}$, where*

$$d_m = \begin{cases} \frac{(2m-1)(2m-2)+4}{6}, & \text{if } 3 \mid m,\\ \frac{(2m-1)(2m-2)}{6}, & \text{if } 3 \nmid m,\end{cases}$$

is the number of effective S_3-orbits of the dihedral group D_{2m}.

In the case of the infinite dihedral group, we got the following:

Lemma 3. *Any element of $pm'_{D_\infty\times D_\infty}(D_\infty)$ is uniquely determined by its values on pairs in the set $\{(a^i, a^j) \mid (i, j) \in \mathbb{N}\times\mathbb{N}\} \cup \{(a^k, a^l b) \mid (k, l) \in \mathbb{N}\times\mathbb{Z}\}$, which can be any values in K^*.*

Theorem 3. *$pM_{D_\infty\times D_\infty}(D_\infty) \simeq (K^*)^{(\mathbb{N}\times\mathbb{N})\times(\mathbb{N}\times\mathbb{Z})}$.*

3 Dicyclic groups

The dicyclic group of order $4n$ is defined by $\mathrm{Dic}_m = \langle a, b \mid a^{2m} = 1,\ b^2 = a^m,\ b^{-1}ab = a^{-1}\rangle$. Among these are the generalized quaternion groups, for which m is a power of 2.

Lemma 4. *The set*

$$S_{\mathrm{Dic}_m} = S_{C_{2m}} \cup \{(a^i, a^k b) \mid 1 \leq i \leq m-1, 0 \leq k \leq 2m-1\} \cup \{(a^m, a^k b) \mid 0 \leq k \leq m-1\}$$

contains exactly one representative of each effective orbit of S_3 on $\mathrm{Dic}_m \times \mathrm{Dic}_m$.

Proposition 1. *Let $G = \mathrm{Dic}_m$, for some natural number $m \geq 2$. If $\sigma \in pm'_{G\times G}(G)$ and $\sigma \sim 1$ then σ is uniquely determined by its values on the pairs*

$$(a, a^k b), \text{ where } 0 \leq k \leq m,$$
$$(a^i, b), \text{ where } 2 \leq i \leq m-1,$$

which can be chosen in K^* arbitrarily, and also by its value on (a^m, b), which must satisfy

$$\sigma^2(a^m, b) = \frac{(\sigma_m(a, b))^2}{(\sigma(a, b)\sigma(a, a^m b))^m}.$$

Theorem 4. *If $G = \mathrm{Dic}_m$, then $pM_{G\times G}(G) \simeq (K^*)^{dc_m - 2m+1}$, where dc_m is the number of effective S_3-orbits of the group Dic_m, given by:*

$$dc_m = \begin{cases} \frac{(4m-1)(4m-2)+4}{6}, & \text{if } 3 \mid m,\\ \frac{(4m-1)(4m-2)}{6}, & \text{if } 3 \nmid m.\end{cases}$$

4 Infinite cyclic group

On [3, Corollary 6.4], there was a description of $pM_{G\times G}(G)$ for finite cyclic groups G. We obtained an analogous for the infinite cyclic group.

Lemma 5. *Any element of $pm'_{\mathbb{Z}\times\mathbb{Z}}(\mathbb{Z})$ is uniquely determined by its values on pairs $(i, j) \in \mathbb{N}\times\mathbb{N}$. Moreover, these values can be chosen arbitrarily in K^*.*

Proposition 2. *If $\sigma \in pm'_{\mathbb{Z}\times\mathbb{Z}}(\mathbb{Z})$ and $\sigma \sim 1$ then σ is uniquely determined by its values on $\{1\}\times\mathbb{N}$, which can be chosen arbitrarily in K^* .*

Theorem 5. *$pM_{\mathbb{Z}\times\mathbb{Z}}(\mathbb{Z}) \simeq (K^*)^{\mathbb{N}}$.*

5 Direct product of two cyclic groups

Proposition 3. *Let $G = C_m \times C_n$, $m, n \in \mathbb{N}$ and $\sigma \in pm'_{G\times G}(G)$ such that $\sigma \sim 1$. Then σ is uniquely determined by its values on the pairs*

$$(a, a^k b^l), \text{ where } 0 \leq k \leq m-1,\ 1 \leq l \leq \lfloor(n-1)/2\rfloor, \quad (1)$$
$$(a, a^k b^{n/2}), \text{ where } 0 \leq k \leq \lfloor(m-1)/2\rfloor \text{ (if } n \text{ is even)}, \quad (2)$$
$$(a^i, b), \text{ where } 2 \leq i \leq \lfloor m/2\rfloor, \quad (3)$$
$$(b, b^l), \text{ where } 1 \leq l \leq \lfloor(n-1)/2\rfloor \text{ (if } n \geq 3), \quad (4)$$

and these values can be chosen in K^* arbitrarily.

Theorem 6. *If $G = C_m \times C_n$, then $pM_{G\times G}(G) \simeq (K^*)^{c_{m,n} - |Q_1| - |Q_2|}$, where $Q_1 \subseteq S_{C_m\times C_n}$ is the set of pairs given by (1) – (3), and $Q_2 \subseteq S_{C_m\times C_n}$ is given by (4).*

6 For future research...

Determine the total component of the partial Schur multiplier for:

- Free abelian groups $\mathbb{Z}^k$, with $k \geq 2$;
- Direct products of arbitrary groups;
- Semidirect products of finite cyclic groups $C_m \rtimes C_n$, or arbitrary groups;
- Symmetric groups S_n where $n \geq 4$.

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8 References

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